F.2.2. Quaternions

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The quaternion representation of rigid body rotations leads to convenient kinematical expressions involving the Euler symmetric parameters (Sections 12.1 and 16.1). Some important properties of quaternions are summarized in this appendix following the formulation of Hamilton [1866] and Whittaker [1961].

Let the four parameters (q_1, q_2, q_3, q_4) form the components of the *quaternion*, $\boldsymbol{q}$, as follows:

$$\boldsymbol{q} \equiv q_4 + iq_1 + jq_2 + kq_3 \quad \text{(Fweb-96)}$$

where i, j, and k are the hyperimaginary numbers satisfying the conditions

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j \end{aligned} \quad \text{(Fweb -97)}$$

The conjugate or inverse of $\boldsymbol{q}$ is defined as

$$\boldsymbol{q}^* \equiv q_4 - iq_1 - jq_2 - kq_3 \quad \text{(Fweb -98)}$$

The quantity, q_4, is the *real* or *scalar part* of the quaternion and $iq_1 + jq_2 + kq_3$ is the *imaginary* or *vector part.*

A vector in three-dimensional space, $\mathbf{U}$, having components U_1, U_2, U_3 is expressed in quaternion notation as a quaternion with a scalar part of zero,

$$\mathbf{U} = iU_1 + jU_2 + kU_3 \quad \text{(Fweb -99)}$$

If the vector $\mathbf{q}$ corresponds to the vector part of $\boldsymbol{q}$ (i.e., $\mathbf{q} = iq_1 + jq_2 + kq_3$), then an alternative representation of $\boldsymbol{q}$ is

$$\boldsymbol{q} = (q_4, \mathbf{q}) \quad \text{(Fweb -100)}$$

Quaternion multiplication is performed in the same manner as the multiplication of complex numbers or algebraic polynomials, except that the order of operations must be taken into account because Eq. (Fweb -97) is not commutative. As an example, consider the product of two quaternions

$$\boldsymbol{q}'' = \boldsymbol{q}' = \left(q_4 + iq_1 + jq_2 + kq_3\right)\left(q_4' + iq_1' + jq_2' + kq_3'\right) \quad \text{(Fweb -101)}$$

Using Eq. (Fweb -97), this reduces to

$$\begin{aligned} \boldsymbol{q}'' = \boldsymbol{q}\boldsymbol{q}' = &\left(-q_1q_1' - q_2q_2' - q_3q_3' + q_4q_4'\right) \\ &+ i\left(q_1q_4' + q_2q_3' - q_3q_2' + q_4q_1'\right) \\ &+ j\left(-q_1q_3' + q_2q_4' + q_3q_1' + q_4q_2'\right) \\ &+ k\left(q_1q_2' - q_2q_1' + q_3q_4' + q_4q_3'\right) \end{aligned} \quad \text{(Fweb -102)}$$

If $\boldsymbol{q}' = (q'_4, \boldsymbol{q}')$, then Eq. (D-7) can alternatively be expressed as

$$\boldsymbol{q}'' = \boldsymbol{q}\boldsymbol{q}' = \left(q_4q_4' - \mathbf{q}\cdot\mathbf{q}',\ q_4\mathbf{q}' + q_4'\mathbf{q} + \mathbf{q}\times\mathbf{q}'\right) \quad \text{(Fweb -103)}$$

The *length* or *norm* of $\boldsymbol{q}$ is defined as

$$\left|\boldsymbol{q}\right| \equiv \sqrt{\boldsymbol{q}\boldsymbol{q}^*} = \sqrt{\boldsymbol{q}^*\boldsymbol{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} \qquad \text{(Fweb -104)}$$

If a set of four Euler symmetric parameters corresponding to the rigid body rotation defined by the transformation matrix, *A* (Section 12.1), are the components of the quaternion, ***q***, then ***q*** is a representation of the rigid body rotation. If $\boldsymbol{q}'$ corresponds to the rotation matrix *A´*, then the rotation described by the product $A'A$ is equivalent to the rotation described by $\boldsymbol{q}\boldsymbol{q}'$. (Note the inverse order of quaternion multiplication as compared with matrix multiplication.)

The transformation of a vector **U**, corresponding to multiplication by the matrix *A*,

$$\mathbf{U}' = A\mathbf{U} \qquad \text{(Fweb -105)}$$

is effected in quaternion algebra by the operation

$$\mathbf{U}' = \boldsymbol{q}^*\mathbf{U}\boldsymbol{q} \qquad \text{(Fweb -106)}$$

See Section 12.1 for additional properties of quaternions used to represent rigid body rotations.

For computational purposes, it is convenient to express quaternion multiplication in matrix form. Specifically, let the components of ***q*** form a four-vector as follows:

$$\boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \qquad \text{(Fweb -107)}$$

This procedure is analogous to expressing the complex number $c = a + ib$ in the form of the two-vector,

$$\boldsymbol{c} = \begin{bmatrix} a \\ b \end{bmatrix}$$

In matrix form, Eq. (Fweb -102) then becomes

$$\begin{bmatrix} q_1'' \\ q_2'' \\ q_3'' \\ q_4'' \end{bmatrix} = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \qquad \text{(Fweb -108)}$$

Given the quaternion components corresponding to two successive rotations, Eq. (Fweb -108) conveniently gives the quaternion components corresponding to the total rotation.

References

1. Hamilton, Sir W. R., *Elements of Quaternions.* London: Longmans, Green and Co., 1866.
2. Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies.* Cambridge: Cambridge University Press, 1961.